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Perturbing Uniform Asymptotically Stable Nonlinear Systems

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1. INTRODUCTION

Our purpose here is to prove new theorems on the eventual uniform-asymptotic stability (hereafter called EvUAS—see Definition 2.4) of the origin 0 for the ordinary differential equation

$$x' = f(t, x) + g(t, x), \tag{P}$$

given that 0 is EvUAS for the equation

$$x' = f(t, x), \tag{E}$$

and that f and g satisfy certain conditions. We always assume that f and g are at least continuous from $[0, \infty) \times R^d$ to R^d. Assume temporarily that the solutions of (P) are unique but do not assume that the zero function is a solution of (P). In fact EvUAS is a natural generalization of uniform asymptotic stability in which it is not assumed that the zero function is a solution (Lemma 2.7).

One main result (see Theorems 4.4, 5.2, 6.1 and 7.1) is

THEOREM A. *Let* 0 *be* EvUAS *for* (E). *Then* 0 *is* EvUAS *for* (P) *if*

(i) *f is Lipschitz and g is diminishing, or*

(ii) *f is periodic and g is diminishing, or*

(iii) *f is inner product and g is absolutely diminishing, or*

(iv) *f is linear and $g = g_1 + g_2$, where g_1 is absolutely diminishing and $g_2 = o(|x|)$.*

Corollaries 4.5 and 7.4 may be summarized as

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THEOREM B. *Let* 0 *be* EvUAS *for* (E). *Let* f *be Lipschitz or periodic. Then* 0 *is* EvUAS *for*

$$x' = f(t, x) + h(t) \tag{1.1}$$

if and only if h *is diminishing. In fact if* h *is not diminishing, then no solution of* (1.1) *can approach zero as* $t \to \infty$.

Both implications of Theorem B are false (Example 5.7) for inner product f and for linear f. Furthermore (Example 5.3), there exist (exponentially) diminishing functions g and there exist functions f which are both inner product and linear such that 0 is exponentially stable for (E) but not EvUAS for (P).

The third result (Example 8.2) shows that, in Theorem A, the conditions on f cannot be weakened too much.

THEOREM C. *There exists a function* f *which is uniformly continuous and locally Lipschitz on* $[0, \infty) \times R^1$ *such that* 0 *is exponentially stable for* (E) *but not* EvUAS *for* (P) *if* $g(t, x) = \gamma(x)e^{-\beta t}$ *for each* $\beta \geqslant 0$ *and each continuous* γ *satisfying* $\gamma(x) > 0$ for $x > 0$.

The final result (Theorems 4.9, 5.8, and 7.2) which we state here shows results such as those in Theorem A cannot be obtained by assuming that g is, in some sense, small in x.

THEOREM D. *Let* $d \geqslant 2$. *Then for each continuous function* $g(x) \not\equiv 0$, *there exists a continuous function* $f(x)$ *such that* 0 *is* UAS *for* (E) *but not for* (P). *If* g *is Lipschitz, then so is* f.

Theorem A generalizes the following result, obtained in stages by Malkin ([*4*], § 1.8), Vrkoc [*13*], Wexler [*14*], Yoshizawa ([*15*], p. 130), Krasovskii ([*6*], p. 102), LaSalle and Rath [*7*], and Strauss and Yorke [*11*]:

THEOREM. *If* 0 *is* UAS *for* (E), *if* f *is Lipschitz, and if* g *is absolutely diminishing, then* 0 *is* EvUAS *for* (P).

Theorem A also generalizes the following result, obtained in stages by Poincaré, Liapunov, Perron, Coddington and Levinson ([*2*], § 13), Brauer [*1*], and Strauss and Yorke [*11*]:

THEOREM. *Let* A *be a constant matrix. If* 0 *is* UAS *for* $x' = Ax$ *and if* $g = g_1 + g_2$, *where* g_1 *is absolutely diminishing and* $g_2 = o(|x|)$, *then* 0 *is "eventually asymptotically stable" for* (P).

More detail on the contributions of the above authors is given in § 4 and § 6. There do not seem to be any results in the literature for f merely periodic or inner product.

Definitions of the above concepts are given later, mostly in § 2. Roughly speaking, g is *absolutely diminishing* (see Def. 2.18) if, for each $0 < m < 1$,

$$\int_t^{t+1} \sup_{m \leqslant |x| \leqslant 1} |g(s, x)| \, ds \to 0 \qquad \text{as} \quad t \to \infty,$$

and g is *diminishing* (see Def. 2.19) if g is absolutely diminishing or if g is continuous in x uniformly with respect to $t \geqslant 0$ and for each fixed x satisfying $0 < |x| < 1$,

$$\sup_{0 \leqslant u \leqslant 1} \left| \int_t^{t+u} g(s, x) \, ds \right| \to 0 \qquad \text{as} \quad t \to \infty.$$

In a practical problem one may know that 0 is, for example, EvUAS and that f satisfies some property, for example, periodicity. *One may not know any more about the unperturbed system.* It makes sense, therefore, to ask what perturbations preserve the EvUAS of 0 for every periodic f. In other words, one wants conditions on admissible perturbations g which do not depend on a particular f but rather on a certain property of f. This is our approach here. To better understand the relationship between properties of f in (E) and the conditions for admissible perturbations g, we use the apparently new concept of perturbation classes, described in § 2.

Actually, uniqueness is not needed in the previous results and is not explicitly assumed in the subsequent sections. It is sometimes implied as we now see. Assume 0 is EvUAS for (E). In the case where f is Lipschitz, (E) has unique solutions but the zero function need not be a solution of (E). If f is periodic, the zero function is a solution (Lemma 7.3) but (E) need not have unique solutions. If f is linear, (E) has the zero solution and uniqueness; while if f is inner product, (E) need have neither the zero solution nor uniqueness.

Finally, we remark that most of the results given here can be extended without difficulty to the case of global uniform-asymptotic stability.

2. Notation and Definitions

Let R^d denote Euclidean d-space and let $|\cdot|$ denote any d-dimensional norm. We shall use $\|\cdot\|$ for the Euclidean norm and $\langle x, y\rangle$ for the inner product of x and y in R^d, i.e., $\langle x, y\rangle = \sum x_i y_i$. Hence $\|x\|^2 = \langle x, x\rangle$. For $r > 0$ let $S_r = \{x \in R^d : |x| \leqslant r\}$. When a sequence $\{a_n\}$ is subscripted by the letter n, it will be implicitly assumed $n = 1, 2, 3, \ldots$, and $a_n \to a$ will mean $a_n \to a$ as $n \to \infty$. Define

$$\mathcal{F}_C = \{\text{continuous functions } f: [0, \infty) \times R^d \to R^d\}.$$

Consider (E) and (P), when f and g belong to $\mathcal{F}_C$. Denote any such solution of (E) at time t by $x(t; t_0, x_0)$ and any such solution of (P) at time t by $y(t; t_0, x_0)$. The following Gronwall inequality will be useful later.

LEMMA 2.1. *If $r(t)$ and $p(t)$ are continuous for $t \geqslant t_0$, if $c \geqslant 0$ and $b \geqslant 0$, and if*

$$r(t) \leqslant c + \int_{t_0}^{t} [br(s) + p(s)]\, ds \qquad (t \geqslant t_0), \tag{2.1}$$

then

$$r(t) \leqslant c e^{b(t-t_0)} + \int_{t_0}^{t} p(s)\, e^{b(t-s)}\, ds \qquad (t \geqslant t_0).$$

We now turn to the definition of the stabilities that we shall use later. The following definitions are stated for (E). Of course, they apply to (P) as well. If the solutions of (E) are unique to the right, (that is, $x(t; t_0, x_0)$ is uniquely determined by (t_0, x_0) for $t \geqslant t_0$) then there is no ambiguity in the use of $x(t; t_0, x_0)$ below. If not, we demand that the conditions given below be satisfied for *all* solutions passing through (t_0, x_0). Actually we shall never assume enough about (P) to guarantee that its solutions are unique to the right. In § 7 we shall not assume uniqueness for (E). We often denote the origin of R^d by 0.

DEFINITION 2.2. The origin is *eventually uniformly stable* (EvUS) if for every $\epsilon > 0$, there exists $\alpha = \alpha(\epsilon) \geqslant 0$ and $\delta = \delta(\epsilon) > 0$ such that

$$|x(t; t_0, x_0)| < \epsilon \qquad \text{for all} \qquad |x_0| < \delta \qquad \text{and} \qquad t \geqslant t_0 \geqslant \alpha.$$

It is *uniformly stable* (US) if one can choose $\alpha(\epsilon) \equiv 0$.

DEFINITION 2.3. The origin is *eventually uniformly attracting* (EvUA) if there exist $\delta_0 > 0$ and $\alpha_0 \geqslant 0$ and if for every $\epsilon > 0$ there exists $T = T(\epsilon) \geqslant 0$ such that

$$|x(t; t_0, x_0)| < \epsilon \qquad \text{for} \qquad |x_0| < \delta_0, \quad t_0 \geqslant \alpha_0, \qquad \text{and} \qquad t \geqslant t_0 + T.$$

It is *uniformly attracting* (UA) if one can choose $\alpha_0 = 0$.

DEFINITION 2.4. The origin is *eventually uniform-asymptotically stable* (EvUAS) if it is both EvUS and EvUA. It is *uniform-asymptotically stable* (UAS) if it is both US and UA.

The following propositions illustrate the relationship between the "eventuality" of stability and the existence of the zero function $x(t) \equiv 0$ as a unique-to-the-right solution of (E).

LEMMA 2.5. *Let* 0 *be* EvUS. *Then* 0 *is* US *if and only if* $x(t) \equiv 0$ *is a unique-to-the-right solution.*

LEMMA 2.6. *Let* 0 *be* EvUA. *Then* 0 *is* UA *if* $x(t) \equiv 0$ *is a unique-to-the right solution.*

LEMMA 2.7. *Let* 0 *be* EvUAS. *Then* 0 *is* UAS *if and only if* $x(t) \equiv 0$ *is a unique-to-the-right solution.*

Example 2.8. For the scalar equation

$$x' = -2x + e^{-t} \tag{2.2}$$

it follows that

$$x(t; t_0, x_0) = x_0 e^{-2(t-t_0)} + e^{-t} - e^{-(2t-t_0)}.$$

Therefore for $t \geqslant t_0 \geqslant 0$, we have

$$| x(t; t_0, x_0)| \leqslant | x_0 | + 2e^{-t_0}$$

and for $T \geqslant 0$ and $t_0 \geqslant 0$, we have

$$| x(t_0 + T; t_0, x_0)| \leqslant | x_0 | e^{-2T} + e^{-T} + e^{-2T}$$

so that 0 is EvUS and UA for (2.2). However, the zero function is not a solution of (2.2). This shows that the converse of Lemma 2.6 does not hold.

It will be convenient later to have the definitions of stability stated in terms of limits.

LEMMA 2.9. *The origin is* EvUS *for* (E) *if and only if*

$$\lim_{\substack{|x_0| \to 0 \\ t_0 \to \infty}} (\sup_{t \geqslant 0} | x(t + t_0; t_0, x_0)|) = 0.$$

It is EvUA *if and only if there exist* $\delta_0 > 0$ *and* $\alpha_0 \geqslant 0$ *such that*

$$\lim_{t \to \infty} (\sup_{\substack{|x_0| < \delta_0 \\ t_0 \geqslant \alpha_0}} | x(t + t_0; t_0, x_0)|) = 0.$$

In this paper we want to consider those perturbations g which preserve EvUAS for every function f belonging to some preassigned class.

DEFINITION 2.10. Let $\mathcal{F} \subset \mathcal{F}_C$. Define the *perturbation classes* $\mathcal{G} = \mathcal{G}(\mathcal{F})$ and $\mathcal{H} = \mathcal{H}(\mathcal{F})$ by

$$\mathcal{G} = \{g \in \mathcal{F}_C : \forall f \in \mathcal{F}, 0 \text{ is EvUAS for (E)} \Rightarrow 0 \text{ is EvUAS for (P)}\},$$

$$\mathcal{H} = \{h \in \mathcal{G}(\mathcal{F}) : h \text{ is independent of } x\}.$$

It is easy to prove the following properties of $\mathcal{G}$ and $\mathcal{H}$.

LEMMA 2.11. *Let $\mathcal{F}_1 \subset \mathcal{F}_2$. Then $\mathcal{G}(\mathcal{F}_1) \supset \mathcal{G}(\mathcal{F}_2)$ and $\mathcal{H}(\mathcal{F}_1) \supset \mathcal{H}(\mathcal{F}_2)$.*

LEMMA 2.12. $\mathcal{G}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \mathcal{G}(\{f\})$.

LEMMA 2.13. *Let $\mathcal{F}$ be closed under addition. Let g_1 and g_2 belong to $\mathcal{F} \cap \mathcal{G}(\mathcal{F})$. Then $g_1 + g_2$ belongs to $\mathcal{G}(\mathcal{F})$.*

It turns out that for the classes $\mathcal{F}$ that we consider here, the perturbation classes $\mathcal{G}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F})$ contain the "diminishing functions" and apparently little else of interest.

DEFINITION 2.14. Let $h : [0, \infty) \to R^d$ be continuous. Then h is *absolutely diminishing* if

$$\int_t^{t+1} | h(s)| \, ds \to 0 \qquad \text{as} \qquad t \to \infty. \tag{2.4}$$

For example if $| h(t)| \to 0$ as $t \to \infty$ or if

$$\int_0^\infty | h(t)|^p \, dt < \infty \qquad \text{for some} \quad p \geqslant 1, \tag{2.5}$$

then h is absolutely diminishing. Relation (2.4) follows from (2.5) by Hölder's inequality, since

$$\int_t^{t+1} | h(s)| \, ds \leqslant \left[\int_t^{t+1} | h(s)|^p \, ds\right]^{1/p} \to 0 \qquad \text{as} \qquad t \to \infty.$$

DEFINITION 2.15. Let $h : [0, \infty) \to R^d$ be continuous. Then h is *diminishing* if

$$\sup_{0 \leqslant u \leqslant 1} \left| \int_t^{t+u} h(s) \, ds \right| \to 0 \qquad \text{as} \qquad t \to \infty. \tag{2.6}$$

Of course if h is absolutely diminishing then h is diminishing. The function

$$\eta(t) = (t \sin t^3, t \cos t^3, 0,..., 0)$$

satisfies (2.6) but not (2.4) because $\| \eta(t)\| = t$. Thus a diminishing function need not be absolutely diminishing. In fact, the function $t^{-1}\eta(t)$ is bounded, diminishing, and not absolutely diminishing. More details are given here in Example 5.3 and also in ([11], Example 4.5).

To extend Definitions 2.14 and 2.15 to functions which depend on x, we need the following.

DEFINITION 2.16. Let $\mathcal{K}_0$ be the class of monotonic, non-negative functions $b(\cdot)$ defined on $[0, \infty)$ such that $b(\rho) \to 0$ as $\rho \to +0$.

DEFINITION 2.17. Let $\mathcal{K}_\infty$ be the class of monotonic, non-negative functions $c(\cdot)$ defined on $[0, \infty)$ such that $c(t) \to 0$ as $t \to \infty$.

DEFINITION 2.18. Let $g \in \mathcal{F}_C$. Then g is *absolutely diminishing* if for some $r > 0$ and every m satisfying $0 < m < r$, there exists an absolutely diminishing function h_m such that

$$| g(t, x)| \leqslant h_m(t) \qquad \text{for all} \qquad t \geqslant 0 \qquad \text{and} \qquad m \leqslant | x | \leqslant r.$$

Coppel ([3], p. 97), Hartman ([5], p. 286), Krasovskii ([6], p. 101), LaSalle and Rath [7], Miller [9], and possibly others have used functions satisfying Definition 2.18 for $m = 0$ in perturbation theorems. The example $g(t, x) = t(t^2 x^2 + 1)^{-1}$ shows that an absolutely diminishing function need not satisfy Definition 2.18 for $m = 0$, because $g(t, 0) = t$ but if $0 < m \leqslant | x | \leqslant r$, then

$$| g(t, x)| \leqslant t(t^2m^2 + 1)^{-1} \to 0 \qquad \text{as} \qquad t \to \infty.$$

DEFINITION 2.19. Let $g \in \mathcal{F}_C$. Then g is *diminishing* if for some $r > 0$ and each fixed x satisfying $0 < | x | \leqslant r$,

$$\sup_{0 \leqslant u \leqslant 1} \left| \int_t^{t+u} g(s, x)\, ds \right| \to 0 \qquad \text{as} \qquad t \to \infty, \tag{2.7}$$

and furthermore if for some $b \in \mathcal{K}_0$ and for each m satisfying $0 < m < r$ there exists an absolutely diminishing function h_m such that for every $t \geqslant 0$, $m \leqslant | x | \leqslant r$, and $m \leqslant | y | \leqslant r$,

$$| g(t, x) - g(t, y)| \leqslant b(| x - y |) + h_m(t). \tag{2.8}$$

If g is absolutely diminishing then g is diminishing, because we may choose $b \equiv 0$ and use $| g(t, x) - g(t, y)| \leqslant | g(t, x)| + | g(t, y)|$. Note that (2.8) is satisfied by any function g which is continuous in x uniformly with respect to t for $t \geqslant 0$ (choose $h_m \equiv 0$), and (2.7) is satisfied if $g(\cdot, x)$ is a diminishing function of t for each constant x satisfying $0 < | x | \leqslant r$. A function g which is diminishing but not absolutely diminishing is given by $g(t, x) = B(t)\, k(x)$, where each column of the matrix B is bounded and diminishing but not necessarily absolutely diminishing and where $k : R^d \to R^d$ is continuous (see also Corollary 4.6). Of course $k(x) \equiv 1$ is allowed since it is not assumed that $k(0) = 0$. For dimension $d = 1$, an example of a diminishing function is $g(t, x) + \sum_{i=1}^{N} b_i(t)\, k_i(x)$, where N is any integer, g is absolutely diminishing, k_i and b_i are continuous real valued functions, and b_i are bounded and diminishing. Although (2.7) only requires that g is diminishing for each constant x, the next lemma shows that (2.8) implies that g is "uniformly" diminishing for x in each annulus.

The proof is a straightforward indirect argument, using (2.8) and (2.7).

LEMMA 2.20. *Let g be diminishing. Then for every m satisfying $0 < m < r$, it follows that*

$$\sup_{\substack{0 \leqslant u \leqslant 1 \\ m \leqslant |x| \leqslant r}} \left| \int_t^{t+u} g(s, x)\, ds \right| \to 0 \qquad as \qquad t \to \infty.$$

3. MAIN LEMMAS

The purpose of this section is to present two lemmas upon which the subsequent results will be based. The hard parts of the proofs of most of the later results are contained in the proof of Lemma 3.1. In § 4 and 5 we shall use Lemma 3.1 with $\lambda \equiv 0$. In § 6 we shall use Lemma 3.1 with $\lambda(\rho) = K\rho$ and $\nu(s) = e^{-\beta s}$. In § 7 we shall use Lemma 3.2, which is a restatement of Lemma 3.1 with $\lambda \equiv 0$. In this section no uniqueness assumptions are made on either (E) or (P).

The first lemma says that if each solution y of (P) is "near" a solution x of an equation (E) for which 0 is EvUAS, then 0 is EvUAS for (P) also. Lemma 3.1 only makes assumptions on the solutions and not the right hand sides of (E) and (P). The main theorems of this paper make various assumptions on (E) and (P) and it is proved that the hypotheses of Lemma 3.1 are valid.

LEMMA 3.1. *Let* 0 *be* EvUAS *for* (E). *Suppose there exist $\lambda \in \mathcal{K}_0$ and $\nu \in \mathcal{K}_\infty$ and for some $r > 0$, each $m \in (0, r)$, and each $\tau > 0$, there exists $F_{\tau,m} \in \mathcal{K}_\infty$ such that for each $\tau_1 \in (0, \tau]$, each $t_0 \geqslant \alpha(r)$ and each solution $y(\cdot)$ of* (P) *satisfying*

$$|y(t_0)| < \delta(r) \qquad and \qquad m \leqslant |y(t)| \leqslant r \qquad for \qquad t_0 \leqslant t \leqslant t_0 + \tau_1,$$

there exists a solution $x(\cdot)$ of (E) *such that*

$$|x(t) - y(t)| \leqslant \lambda(|y(t_0)|)\, \nu(t - t_0) + F_{\tau,m}(t_0) \tag{3.1}$$

for all $t_0 \leqslant t \leqslant t_0 + \tau_1$. Then 0 *is* EvUAS *for* (P).

Remark. In the above statement $\alpha(r)$ and $\delta(r)$ come from Definition 2.2.

Proof. We are given $\alpha(\cdot)$, $\delta(\cdot)$, $\alpha_0 \geqslant 0$, $\delta_0 > 0$, and $T(\cdot)$ since 0 is EvUAS for (E). We shall produce $\alpha^p(\cdot)$, $\delta^p(\cdot)$, $\alpha_0{}^p \geqslant 0$, $\delta_0{}^p > 0$, and $T^p(\cdot)$ which show that 0 is EvUAS for (P). We do so by proving four claims.

We may assume without loss of generality that

$$\delta_0 \leqslant \delta(r) \leqslant r \quad \text{and} \quad \alpha_0 \leqslant \alpha(r). \tag{3.2}$$

Choose $\gamma = \gamma(\delta_0) > 0$ such that

$$\lambda(\gamma)\, \nu(0) < \delta_0/4. \tag{3.3}$$

Let $\epsilon > 0$. We may also assume without loss that

$$\epsilon < \gamma \leqslant \delta_0/2. \tag{3.4}$$

Choose $\delta^p = \delta^p(\epsilon)$ so that

$$\delta^p(\epsilon) < \delta(\epsilon/2)/4 \leqslant \epsilon/8, \tag{3.5}$$

$$\lambda(\delta^p)\, \nu(0) < \delta(\epsilon/2)/2 \leqslant \epsilon/4. \tag{3.6}$$

Then choose $\tau = \tau(\epsilon)$ so that

$$\tau(\epsilon) \geqslant T(\delta^p/2), \tag{3.7}$$

$$\lambda(\gamma)\, \nu(\tau) < \delta^p/4. \tag{3.8}$$

Finally choose $\alpha^p = \alpha^p(\epsilon)$ so that

$$\alpha^p(\epsilon) \geqslant \alpha(\epsilon/2) \geqslant \alpha(r), \tag{3.9}$$

$$F_{\tau,\delta^p}(\alpha^p) < \delta^p/4. \tag{3.10}$$

Claim 1. Let $t_1 \geqslant \alpha^p(\epsilon)$. Then, for every solution $y(\cdot)$ of (P),

$$|\, y(t_1)| \leqslant \delta^p(\epsilon) \quad \text{implies} \quad |\, y(t)| < \epsilon \quad \text{for all} \quad t_1 \leqslant t \leqslant t_1 + \tau.$$

Proof. Suppose not for some solution $y(\cdot)$. Let t_3 be the first point such that $|\, y(t_3)| = \epsilon$ and let $t_2 < t_3$ be the last point such that $|\, y(t_2)| = \delta^p$. Let $\tau_1 = t_3 - t_2 \leqslant \tau$. Then for all $t_2 \leqslant t \leqslant t_2 + \tau_1$, $\delta^p \leqslant |\, y(t)| \leqslant \epsilon \leqslant r$. Since $|\, y(t_2)| = \delta^p(\epsilon) < \delta(r)$ and $t_2 \geqslant \alpha^p(\epsilon) \geqslant \alpha(r)$, we may select a solution $x(\cdot)$ of (E) so that (3.1) holds on the interval $[t_2, t_2 + \tau_1]$. Then

$$\begin{aligned}
|\, x(t) - y(t)| &\leqslant \lambda(|\, y(t_2)|)\, \nu(t - t_2) + F_{\tau,\delta^p}(t_2) \\
&\leqslant \lambda(\delta^p)\, \nu(0) + F_{\tau,\delta^p}(\alpha^p) \\
&< \delta(\epsilon/2)/2 + \delta^p/4
\end{aligned}$$

for all $t_2 \leqslant t \leqslant t_2 + \tau_1$, using (3.6) and (3.10). Thus, using (3.5), we have

$$\begin{aligned} |x(t_2)| &\leqslant |x(t_2) - y(t_2)| + |y(t_2)| \\ &< \delta(\epsilon/2)/2 + \delta^p/4 + \delta^p < \delta(\epsilon/2). \end{aligned}$$

This and (3.9) imply that $|x(t)| < \epsilon/2$ for all $t \geqslant t_2$. Therefore, using (3.5),

$$\begin{aligned} |y(t)| &\leqslant |y(t) - x(t)| + |x(t)| \\ &< \delta(\epsilon/2)/2 + \delta^p/4 + \epsilon/2 < \epsilon \end{aligned}$$

for all $t_2 \leqslant t \leqslant t_2 + \tau_1$, which is a contradiction at $t = t_3 = t_2 + \tau_1$. This proves Claim 1.

Claim 2. Let $t_4 \geqslant \alpha^p(\epsilon)$. If $y(\cdot)$ is any solution of (P) satisfying

$$|y(t)| < \gamma \qquad \text{for all} \qquad t_4 \leqslant t \leqslant t_4 + \tau,$$

then there exists t_5 such that

$$|y(t_5)| < \delta^p(\epsilon) \qquad \text{and} \qquad t_4 \leqslant t_5 \leqslant t_4 + \tau.$$

Proof. Suppose not for some solution $y(\cdot)$. Then $\delta^p(\epsilon) \leqslant |y(t)| < \gamma \leqslant r$ for all $t_4 \leqslant t \leqslant t_4 + \tau$. Since $|y(t_4)| < \gamma \leqslant \delta(r)$ and $t_4 \geqslant \alpha^p(\epsilon) \geqslant \alpha(r)$, we may select a solution $x(\cdot)$ of (E) so that (3.1) holds on the interval $[t_4, t_4 + \tau]$. Then

$$\begin{aligned} |x(t_4)| &\leqslant |x(t_4) - y(t_4)| + |y(t_4)| \\ &\leqslant \lambda(\gamma)\, \nu(0) + F_{\tau,\delta^p}(\alpha^p) + \gamma \\ &< \delta_0/_4 + \delta^p/4 + \delta_0/2 < \delta_0, \end{aligned}$$

where we have used (3.3), (3.10), (3.4), and (3.5). Furthermore, $t_4 \geqslant \alpha(r) \geqslant \alpha_0$. Thus, using (3.8), (3.10), and (3.7), we have

$$\begin{aligned} |y(t_4 + \tau)| &\leqslant |y(t_4 + \tau) - x(t_4 + \tau)| + |x(t_4 + \tau)| \\ &\leqslant \lambda(|y(t_4)|)\, \nu(\tau) + F_{\tau,\delta^p}(t_4) + \delta^p/2 \\ &< \delta^p/4 + \delta^p/4 + \delta^p/2 = \delta^p(\epsilon). \end{aligned}$$

This contradiction proves Claim 2.

Claim 3. The origin is EvUS for (P).

Proof. Let $t_0 \geqslant \alpha^p(\epsilon)$ and $|x_0| < \delta^p(\epsilon)$. Suppose there were a solution $y(t; t_0, x_0)$ which is not bounded by ϵ for $t \geqslant t_0$. Let s_2 be the first point such that $|y(s_2; t_0, x_0)| = \epsilon$ and let $s_1 < s_2$ be the last point such that $|y(s_1; t_0, x_0)| = \delta^p(\epsilon)$. By Claim 1, since $s_1 \geqslant \alpha^p(\epsilon)$, we would have

$$s_2 > s_1 + \tau. \tag{3.11}$$

By Claim 2 applied to the interval $[s_1, s_2]$ on which

$$\delta^p(\epsilon) \leqslant |y(t; t_0, x_0)| \leqslant \epsilon < \gamma,$$

we would have $s_2 - s_1 < \tau$, that is, $s_2 < s_1 + \tau$, contradicting (3.11) and proving Claim 3.

Claim 4. The origin is EvUA for (P).

Proof. Choose $\delta_0^p = \delta^p(\gamma)$ and $\alpha_0^p = \alpha^p(\gamma)$. Choose

$$T^p(\epsilon) = \tau + \alpha^p(\epsilon).$$

Let $t_0 \geqslant \alpha_0^p$ and $|x_0| < \delta_0^p$. Then $|y(t; t_0, x_0)| < \gamma$ for all $t \geqslant t_0$. By Claim 2 applied to the interval $[\alpha^p + t_0, \alpha^p + t_0 + \tau]$, there exists s_3 such that

$$\alpha^p(\epsilon) + t_0 \leqslant s_3 \leqslant \alpha^p(\epsilon) + t_0 + \tau$$

and $|y(s_3; t_0, x_0)| < \delta^p(\epsilon)$. By Claim 3, $|y(t; t_0, x_0)| < \epsilon$ for all $t \geqslant s_3$, and hence *a fortiori* for all

$$t \geqslant \alpha^p(\epsilon) + \tau(\epsilon) + t_0 = t_0 + T^p(\epsilon),$$

proving Claim 4 and Lemma 3.1.

The following result is actually a special case of Lemma 3.1 (with $\lambda \equiv 0$), but expressed in a different form.

LEMMA 3.2. *Let* 0 *be* EvUAS *for* (E). *Suppose that for some $r > 0$, each $m \in (0, r)$, each $\tau > 0$, and each $\epsilon > 0$, there exists $\beta = \beta(m, \epsilon, \tau) > 0$ such that for each $t_0 \geqslant \beta$, each $\tau_1 \in (0, \tau]$, and each solution $y(\cdot)$ of (P) satisfying*

$$|y(t_0)| < \delta(r) \quad \text{and} \quad m \leqslant |y(t)| \leqslant r \quad \text{for} \quad t_0 \leqslant t \leqslant t_0 + \tau_1,$$

there is a solution $x(\cdot)$ of (E) *satisfying*

$$|x(t) - y(t)| < \epsilon \quad \text{for} \quad t_0 \leqslant t \leqslant t_0 + \tau_1 .$$

Then 0 *is* EvUAS *for* (P).

4. LIPSCHITZ FUNCTIONS

We say that f is a *Lipschitz* function if there exist $r > 0$ and $L > 0$ such that $|f(t, x) - f(t, y)| \leqslant L|x - y|$ for all $t \geqslant 0$ and all x and y in S_r. Define

$$\mathcal{F}_{\text{Lip}} = \{f \in \mathcal{F}_G : f \text{ is Lipschitz}\}.$$

For $f \in \mathcal{F}_{\text{Lip}}$ it need not happen that $f(t, 0) = 0$. Nevertheless, if 0 is to be EvUAS for (E), $f(t, 0)$ cannot be completely arbitrary. We begin with such a necessary condition on f in order that 0 be EvUAS.

LEMMA 4.1. *Let $f \in \mathcal{F}_{\mathrm{Lip}}$ and let* 0 *be* EvUAS *for* (E). *Then $f(t, 0)$ is diminishing.*

Proof. Let $x(\cdot)$ be a solution of (E) such that $|x(t)| \to 0$ as $t \to \infty$. Let f have Lipschitz constant L on $[0, \infty) \times S_r$ for some r. Then if $0 \leqslant u \leqslant 1$ and if T is chosen so that $|x(t)| \leqslant r$ for $t \geqslant T$,

$$x(t+u) - x(t) = \int_t^{t+u} [f(s, x(s)) - f(s, 0)]\, ds + \int_t^{t+u} f(s, 0)\, ds.$$

Therefore $t \geqslant T$

$$\sup_{0 \leqslant u \leqslant 1} \left| \int_t^{t+u} f(s, 0)\, ds \right| \leqslant (2 + L) \sup_{0 \leqslant u \leqslant 1} |x(t+u)| \to 0 \quad \text{as} \quad t \to \infty.$$

Thus $f(t, 0)$ is diminishing.

LEMMA 4.2. *Let $f \in \mathcal{F}_{\mathrm{Lip}}$ and let* 0 *be* EvUAS *for* (E). *Let $g \in \mathcal{F}_C$. If $y(\cdot)$ is any solution of* (P) *such that $|y(t)| \to 0$ as $t \to \infty$, then the function $h(t) \equiv g(t, y(t))$ is diminishing.*

Proof. Let $|y(t)| \to 0$ as $t \to \infty$. For large t and for $0 \leqslant u \leqslant 1$,

$$\left| \int_t^{t+u} g(s, y(s))\, ds \right| \leqslant |y(t+u)| + |y(t)| + L \sup_{0 \leqslant v \leqslant 1} |y(t+v)| + \left| \int_t^{t+u} f(s, 0)\, ds \right|.$$

The result follows by Lemma 4.1.

COROLLARY 4.3. $\mathcal{H}(\mathcal{F}_{\mathrm{Lip}}) \subset \{h(t) : h \textit{ is diminishing}\}$.

THEOREM 4.4. *For the class $\mathcal{F}_{\mathrm{Lip}}$*

$$\mathcal{G}(\mathcal{F}_{\mathrm{Lip}}) \supset \{g(t, x) : g \textit{ is diminishing}\},$$
$$\mathcal{H}(\mathcal{F}_{\mathrm{Lip}}) = \{h(t) : h \textit{ is diminishing}\}.$$

Remark. A history of this theorem is given at the end of this section.

Proof. By Corollary 4.3, we need only show that $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$ contains the diminishing functions. Let $f \in \mathcal{F}_{\mathrm{Lip}}$ and let g be diminishing. Let $r > 0$ be given by Definition 2.19 and shrink r if necessary so that f has Lipschitz constant L on $[0, \infty) \times S_r$. Let $0 < m < r$. Define

$$E_m{}^*(t) = \sup \left| \int_t^{t+u} g(s, y(s))\, ds \right|,$$

where the supremum is evaluated with respect to all solutions $y(\cdot)$ of (P) satisfying

$$m \leqslant |y(s)| \leqslant r \qquad \text{for} \qquad t \leqslant s \leqslant t + u$$

and then with respect to all $0 \leqslant u \leqslant 1$. We shall first prove that $E_m{}^*(t) \to 0$ as $t \to \infty$. Then we shall use Gronwall's inequality to arrive at an inequality of the type (3.1). We shall then apply Lemma 3.1 to obtain the result.

Suppose $E_m{}^*(t) \not\to 0$. Then there exist $\epsilon > 0$, sequences $\{u_n\}$ and $\{t_n\}$, and a sequence of solutions $\{y_n(\cdot)\}$ of (P) on $[t_n, t_n + u_n]$ such that $0 \leqslant u_n \leqslant 1$, $u_n \to u_0 \in [0, 1]$, $t_n \to \infty$, $m \leqslant |y_n(s)| \leqslant r$ for $t_n \leqslant s \leqslant t_n + u_n$, and

$$\left| \int_{t_n}^{t_n+u_n} g(s, y_n(s))\, ds \right| > \epsilon. \tag{4.1}$$

Let w satisfy $0 < w < 1$, w^{-1} an integer, and

$$b(Lrw + b(2r)\, w + w) < \epsilon, \tag{4.2}$$

where b comes form (2.8). Then

$$\left| \int_{t_n+jwu_n}^{t_n+(j+1)wu_n} g(s, y_n(s))\, ds \right| > \epsilon w \tag{4.3}$$

for some j between 0 and $w^{-1} - 1$, because otherwise

$$\left| \int_{t_n}^{t_n+u_n} g(s, y_n(s))\, ds \right| \leqslant \sum_{j=0}^{w^{-1}-1} \left| \int_{t_n+jwu_n}^{t_n+(j+1)wu_n} g(s, y_n(s))\, ds \right| \leqslant \epsilon,$$

contradicting (4.1). Let $s_n = t_n + jwu_n$ and $v_n = t_n + (j + 1)\, wu_n$ such that (4.3) holds.

For t in $[s_n, v_n]$

$$\begin{aligned} |y_n(t) - y_n(s_n)| &\leqslant \int_{s_n}^t |f(s, y_n(s)) - f(s, 0)|\, ds + \left| \int_{s_n}^t f(s, 0)\, ds \right| \\ &\quad + \int_{s_n}^t |g(s, y_n(s)) - g(s, y_n(s_n))|\, ds + \left| \int_{s_n}^t g(s, y_n(s_n))\, ds \right| \\ &\leqslant Lrwu_n + b(2r)\, wu_n + H_m(t_n), \end{aligned}$$

where

$$H(t) \equiv H_m(t)$$
$$\equiv \sup_{\substack{0 \leqslant u \leqslant 1 \\ m \leqslant |x| \leqslant r}} \left[\left| \int_t^{t+u} f(s, 0)\, ds \right| + \left| \int_t^{t+u} g(s, x)\, ds \right| + \int_t^{t+u} h_m(s)\, ds \right] \to 0 \quad \text{as} \quad t \to \infty,$$

using Lemma 4.1, Lemma 2.20, and (2.8). Then from (4.3)

$$\begin{aligned} \epsilon w &< \int_{s_n}^{v_n} | g(s, y_n(s)) - g(s, y_n(s_n))|\, ds + \left| \int_{s_n}^{v_n} g(s, y_n(s_n))\, ds \right| \\ &\leqslant b\,(\sup_{s_n \leqslant t \leqslant v_n} | y_n(t) - y_n(s_n)|)\, wu_n + H(t_n) \\ &\leqslant b(Lrwu_n + b(2r)\, wu_n + H(t_n))\, wu_n + H(t_n) \end{aligned}$$

for every n. Since $u_n \to u_0 \leqslant 1$ and $H(t_n) \to 0$, and since

$$Lrwu_n + b(2r)\, wu_n + H(t_n) \leqslant Lrw + b(2r)\, w + w$$

for large enough n, we must have $\epsilon w \leqslant b(Lrw + b(2r)\, w + w)\, w$. This is a contradiction to (4.2). Thus $E_m^*(t) \to 0$ as $t \to \infty$.

Define

$$E_m(t) = \sup_{T \geqslant t} E_m^*(T).$$

Then $E_m(t) \downarrow 0$ as $t \to \infty$. Let 0 be EvUAS for (E), so that we are given $\alpha(\cdot)$ and $\delta(\cdot)$. Let $\tau > 0$, let $\tau_1 \in (0, \tau]$, and let $t_0 \geqslant \alpha(r)$. Let $y(\cdot)$ be any solution of (P) satisfying $| y(t_0)| < \delta(r)$ and

$$m \leqslant | y(t)| \leqslant r \qquad \text{for} \qquad t_0 \leqslant t \leqslant t_0 + \tau_1 .$$

Let $x(\cdot)$ be that solution of (E) such that $x(t_0) = y(t_0)$. Then if $t \in [t_0 , t_0 + \tau_1]$ we have $| x(t)| \leqslant r$ and hence

$$\begin{aligned} | x(t) - y(t)| &\leqslant \int_{t_0}^t | f(s, x(s)) - f(s, y(s))|\, ds + \left| \int_{t_0}^t g(s, y(s))\, ds \right| \\ &\leqslant \int_{t_0}^t L| x(s) - y(s)|\, ds + (\tau + 1)\, E_m(t_0). \end{aligned}$$

By Lemma 2.1, $| x(t) - y(t)| \leqslant (\tau + 1)\, e^{L\tau} E_m(t_0)$.
Since m was arbitrary, the hypotheses of Lemma 3.1 hold with $\lambda \equiv 0$ and

$$F_{\tau,m} = (\tau + 1)\, e^{L\tau} E_m$$

Therefore 0 is EvUAS for (P); hence $g \in \mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. The proof is now complete.

Corollary 4.5 restates the second part of Theorem 4.4.

Corollary 4.5. *If $f \in \mathcal{F}_{\mathrm{Lip}}$ and 0 is* EvUAS *for* (E), *then 0 is* EvUAS *for*

$$y' = f(t, y) + h(t) \tag{4.4}$$

if and only if h is diminishing.

Corollary 4.6. *Let $B(t)$ be a bounded matrix on $[0, \infty)$ whose columns are diminishing. Let $k : R^d \to R^d$ be continuous. Then $B(t)\, k(x) \in \mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$.*

Remarks. For functions satisfying (2.8) (even with $h_m \equiv 0$), we do not know whether (2.7) is also necessary for g to belong to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. For functions satisfying (2.7), we do not know whether (2.8) is necessary for g to belong to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. More specifically, although we have given in Corollary 4.5 a necessary and sufficient condition for $h(t)$ to belong to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$, we have not been able to find a necessary and sufficient condition on $B(t)$ in order that $B(t)x$ belong to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. "The columns of $B(t)$ are diminishing" is not such a condition because of Example 4.7 below.

Example 4.7. We show that if $B(\cdot)$ is diminishing but not bounded, then $B(t)x$ need not belong to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. We consider on $[1, \infty)$ the scalar equation

$$y' = -y + h(t) + B(t)\, y, \tag{4.5}$$

where $h(t) = t^2 \sin t^4$ and $B(t) = (t^{-2} \sin t^4)'$.
Since h is diminishing, 0 is EvUAS for $x' = -x + h(t)$ by Theorem 4.4. Suppose $B(t)\, y$ belongs to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. Then 0 is EvUAS for (4.5). Let $y(\cdot)$ be a solution of (4.5) which tends to zero as $t \to \infty$. By Lemma 4.2, $B(t)\, y(t)$ must be diminishing. But

$$\int_t^\tau B(s)\, y(s)\, ds = [e^{\int_t^\tau B(s)ds} - 1]\, y(t) + \int_t^\tau [e^{\int_s^\tau B(\sigma)d\sigma} - 1][h(s) - y(s)]\, ds,$$

because differentiating both sides with respect to τ results in the variation of of constants formula for $x' = B(s)\, x + [h(s) - y(s)]$, multiplied through by $B(\tau)$. Since $B(t)$ is diminishing, $B(t)\, y(t)$ is diminishing, and $|\, y(t)| \to 0$, we must have

$$\sup_{t \leqslant \tau \leqslant t+1} \int_t^\tau h(s)\, e^{\int_s^\tau B(\sigma)d\sigma}\, ds \to 0 \qquad \text{as} \qquad t \to \infty. \tag{4.6}$$

In fact (4.6) is necessary for 0 to be EvUAS for any scalar equation of the form (4.5) in which h is diminishing. In this example

$$\int_s^\tau B(\sigma)\, d\sigma = \tau^{-2} \sin \tau^4 - s^{-2} \sin s^4.$$

Therefore

$$\int_t^\tau h(s)\, e^{\int_s^\tau B(\sigma)d\sigma}\, ds = \exp(\tau^{-2}\sin\tau^4)\int_t^\tau s^2 \sin s^4 \exp(-s^{-2}\sin s^4)\, ds$$

$$= \exp(\tau^{-2}\sin\tau^4)\int_t^\tau [s^2\sin s^4 - \sin^2 s^4 + \tfrac{1}{2}(s^2\sin s^4)(\xi^{-4}\sin^2\xi^4)]\, ds,$$

where ξ is between t and τ. Now $\exp(\tau^{-2}\sin\tau^4) \to 1$ as $\tau \to \infty$. The suprema for $\tau \in [t, t+1]$ of the integrals first and third terms in the above integrand tend to zero as $t \to \infty$. The supremum for $\tau \in [t, t+1]$ of the integral of the second term does not. Thus (4.6) does not hold. This is a contradiction to the supposition that $B(t)y \in \mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. Therefore $B(t)y \notin \mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$ for this example.

A corollary of Theorem 4.4 is: *if* $f \in \mathcal{F}_{\mathrm{Lip}}$, 0 *is* UAS *for* (E), *and* $h(t)$ *is diminishing, then* 0 *is* EvUAS *for* (4.4). We can now prove a converse of this result.

THEOREM 4.8. *Let* $f \in \mathcal{F}_{\mathrm{Lip}}$ *and let* 0 *be* EvUAS *for* (E). *Then there exist* $f_1 \in \mathcal{F}_{\mathrm{Lip}}$ *and a diminishing function* $h(t)$ *such that* $f(t, x) = f_1(t, x) + h(t)$ *and* 0 *is* UAS *for* $x' = f_1(t, x)$.

Proof. By Lemma 4.1, $h(t) = f(t, 0)$ is diminishing. Let

$$f_1(t, x) = f(t, x) - f(t, 0).$$

Then 0 is EvUAS for $x' = f_1(t, x)$ by Theorem 4.4, and therefore it is UAS by Lemma 2.7. The result is proved.

If the dimension of the system (E) is one, then $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$ contains functions which are independent of t, for example, $g(t, x) = -x$. But if $d \geqslant 2$, then $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$ contains no non-trivial, Lipschitz functions independent of t, in marked contrast to the situation for total stability. We say that a function g of x alone is *trivial* if it vanishes identically in S_r for some $r > 0$.

THEOREM 4.9. *If* $d \geqslant 2$, *then* $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$ *contains no non-trivial, Lipschitz function which is independent of t.*

Remark. The idea of the proof is as follows: imagine that $g(x) = -x$ for $d = 2$. We choose a sequence $x_n \to 0$ and a disjoint sequence of disks C_n with center x_n. We define $f(x_n) = x_n$ and $f(x) = -x$ outside any disk. Thus $f(x_n) + g(x_n) = 0$, hence 0 cannot be UAS for (P). The problem is now to define f inside the disks so that it is Lipschitz and so that 0 is UAS for (E). This means we cannot allow a solution to remain forever within a disk, nor can we allow solutions to "re-enter" a disk. We call this construction the "pinball machine example."

Proof. Let $g(x)$ be any nontrivial Lipschitz function, i.e., there exist $x_n \in R^d$ with $|x_n| \to 0$ and $g(x_n) \neq 0$. We may assume without loss that $|x_n| > 4|x_{n+1}|$. Define $\beta_n = |x_n|/4$. Let y_n be defined as follows: if $g(x_n)$ is not parallel to x_n, choose $y_n = 0$. If $g(x_n)$ is parallel to x_n, choose y_n so that

$$\langle y_n, x_n \rangle = 0 \qquad \text{and} \qquad 0 < |y_n| < |x_n|.$$

This is possible since $d \geqslant 2$. Note that for any $x \in R^d$ $\|x - x_n\| < 2\beta_n$ for at most one n. Hence if we define

$$B_{n,1} = \{x : \|x - x_n\| \leqslant \beta_n\},$$

$$B_{n,2} = \{x : \|x - x_n\| \leqslant 2\beta_n\},$$

and

$$A_n = \overline{B_{n,2} - B_{n,1}},$$

then $B_{n,2} \cap B_{m,2}$ is empty for $m \neq n$. Define

$$f(x) = \begin{cases} -x & \text{for } x \notin \bigcup_{n=1}^{\infty} B_{n,2}, \\ -(\rho - 1)x + (\rho - 2)x_n & \text{for } x \in A_n, \\ -x_n\rho + g(x_n)(\rho - 1) + \rho(\rho - 1)y_n & \text{for } x \in B_{n,1}, \end{cases}$$

where $\rho = \|x - x_n\|/\beta_n$. Then $f \in \mathscr{F}_{\mathrm{Lip}}$. Since $f(x_n) = -g(x_n)$, the points x_n are rest points for

$$x' = f(x) + g(x), \tag{4.7}$$

hence 0 is not EvUAS for (4.7). We need only prove that 0 is UAS for

$$x' = f(x) \tag{4.8}$$

to complete the proof.

If x is such that $|x| = 2|x_n|$ for some n, then $f(x) = -x$. Thus 0 is US for (4.8). Since (4.8) is autonomous, we now need only show that every solution tends to zero.

Suppose some solution $x(\cdot)$ of (4.8) does not tend to zero. Then

$$x(T) \in B_{N,2} \qquad \text{for some} \qquad T > 0 \qquad \text{and some} \qquad N,$$

otherwise $x(t) \to 0$ exponentially as $t \to \infty$. It is easy to see that if $x(\cdot)$ leaves $B_{N,2}$, it cannot later enter $B_{k,2}$ for $k < N$. Outside of every $B_{n,2}$, the trajectory of $x(\cdot)$ is a straight line. Since $B_{N,2}$ is convex, if $x(\cdot)$ leaves $B_{N,2}$, it cannot later enter $B_{N,2}$. Thus if $x(\cdot)$ leaves $B_{N,2}$ and does not tend to zero, it must enter $B_{k,2}$ for some $k > N$. There must exist N and T such that

$$x(t) \in B_{N,2} \qquad \text{for all} \qquad t \geqslant T.$$

Let H be the hyperplane through the point x_N normal to the vector x_N. Suppose $x(\cdot)$ never enters the ball $B_{N,1}$. Then for all $t \geqslant T$, $x(t) \in A_N$. But in A_N, $\langle x'(t), x_N \rangle$ is negative and bounded away from zero. Hence $x(t)$ cannot remain in A_N for all $t \geqslant T$, a contradiction. This shows that

$$x(T_1) \in B_{N,1} \qquad \text{for some} \qquad T_1 .$$

On the surface of $B_{N,1}$, $f(x) = -x_N$. Thus if $x(\cdot)$ were to leave $B_{N,1}$, it would have to leave on the side of H containing the origin. From there, $x(\cdot)$ could not cross $H \cap A_N$. Thus $x(\cdot)$ could not re-enter $B_{N,1}$. Therefore, $x(\cdot)$ would remain in A_N for all future time, which was shown above to be impossible. This shows that

$$x(t) \in B_{N,1} \qquad \text{for all} \qquad t \geqslant T_1 .$$

Let Γ be the positive limit set of $x(\cdot)$. Then $\Gamma \subset B_{N,1}$ and Γ is invariant. Now either $g(x_N)$ and x_N are parallel, or they are not. Suppose first that $g(x_N)$ and x_N are parallel. Consider the "Liapunov function"

$$V(x) = \langle x, y_N \rangle.$$

In $B_{N,1}$ V is C^1 and

$$\dot{V}(x) = \langle f(x), y_N \rangle = \rho(\rho - 1) \| y_N \|^2 \leqslant 0,$$

since $\langle x_N , y_N \rangle = \langle g(x_N), y_N \rangle = 0$. We thus have that (see Lemma 5 of [*10*], for example)

$$\Gamma \subset \{x : \dot{V}(x) = 0\}. \tag{4.9}$$

But $\{x : \dot{V}(x) = 0\} = \{x_N\} \cup \{x : \| x - x_N \| = \beta_N\}$ contains no invariant sets, a contradiction.

Now suppose that $g(x_N)$ and x_n are not parallel. Then $y_N = 0$. Let z_N be the component of $g(x_N)$ perpendicular to x_N, i.e.,

$$z_N = g(x_N) - \| x_N \|^{-2} x_N \langle g(x_N), x_N \rangle.$$

Consider the "Liapunov function"

$$V(x) = \langle x, z_N \rangle.$$

In $B_{N,1}$, V is C^1 and using twice the fact that $\langle x_N , z_N \rangle = 0$, we obtain

$$\begin{aligned} \dot{V}(x) &= \langle f(x), z_N \rangle = (\rho - 1) \langle g(x_N), z_N \rangle \\ &= (\rho - 1) \left\langle g(x_N) - \frac{x_N}{\| x_N \|^2} \langle g(x_N), x_N \rangle, z_N \right\rangle = (\rho - 1) \| z_N \|^2 \leqslant 0. \end{aligned}$$

Since $g(x_N)$ and x_N are not parallel,

$$\{x : \dot{V}(x) = 0\} = \{x : \| x - x_N \| = \beta_N\},$$

which has no invariant subsets. This contradicts (4.9).

Both of these contradictions violate the supposition that $x(t)$ does not tend to zero as $t \to \infty$. The theorem is proved.

Remark. Theorem 4.4 has a long history. Malkin showed that $\mathcal{G}(\mathcal{F}_{\text{Lip}})$ contains the Lipschitz functions g_1 satisfying

$$\sup_{|x| \leqslant r} | g_1(t, x)| \to 0 \qquad \text{as} \qquad t \to \infty.$$

Vrkoc [13] proved that $\mathcal{G}(\mathcal{F}_{\text{Lip}})$ contains the functions g_2 satisfying

$$\int_0^\infty \sup_{|x| \leqslant r} | g_2(t, x)| \, dt < \infty.$$

However both Malkin and Vrkoc used UAS rather than EvUAS for (E), and Malkin assumed that $g_1(t, 0) = 0$. Wexler [14] extended Malkin's result to the EvUAS case. Yoshizawa ([15], p. 130) extended the results of both Malkin and Vrkoc to the EvUAS case in his more general presentation of stability of sets. Krasovskii ([16], p. 102) showed that if 0 is UAS for (E), if g is absolutely diminishing with $m = 0$, and if $g(t, 0) = 0$, then 0 is US for (P). LaSalle and Rath [7] announced the extension of Krasovskii's result to the case where 0 is UAS for (E) and EvUAS for (P), so that $g(t, 0)$ need not vanish. We [11] proved that if 0 is UAS for (E), if $g(t, x)$ is absolutely diminishing, and if $h(t)$ is diminishing then 0 is EvUAS for $x' = f(t, x) + g(t, x) + h(t)$. Theorem 4.4 extends all of these results. The results of Malkin, Vrkoc, and Krasovskii mentioned above are also presented in Halanay's book ([4], § 1.8).

Corollary 4.5 was observed previously in [11]. The other results of this section are new.

5. Inner Product Functions

We say that f is an *inner product* function if there exist $r > 0$ and $L > 0$ such that $\langle x - y, f(t, x) - f(t, y)\rangle \leqslant L \| x - y \|^2$ for all $t \geqslant 0$ and all x and y in S_r. Define

$$\mathcal{F}_{\text{Inn}} = \{f \in \mathcal{F}_C : f \text{ is inner product}\}.$$

Note that if $f \in \mathcal{F}_{\text{Inn}}$, then all solutions of (E) are unique to the right but not necessarily to the left, as shown by Lemma 5.1 and the example $x' = -x^{1/3}$.

Consider $f(t, x) = a(t)x$. If $f \in \mathcal{F}_{\text{Lip}}$, then $|a(t)|$ is bounded. But this does not seem appropriate. No restriction should be made on how negative $a(\cdot)$ can be. If $f \in \mathcal{F}_{\text{Inn}}$ with constant L, then $a(t) \leqslant L$ for all t. For example if $a(t) = -t^2$, then $L = 0$ suffices, though $|a(t)| \to \infty$. These theorems allow us to perturb (E) when, for example, $f(t, x) = -tx^{1/3}$, which is neither Lipschitz, nor linear, nor periodic.

LEMMA 5.1. *Let $f \in \mathcal{F}_{\text{Inn}}$ and let $g(t, x)$ be absolutely diminishing. Let $a \geqslant 0$ and $\tau > 0$. Let $x(\cdot)$ be any solution of* (E) *such that $\| x(t)\| \leqslant r$ for $a \leqslant t \leqslant a + \tau$ and let $y(\cdot)$ be any solution of* (P) *such that*

$$m \leqslant \| y(t)\| \leqslant r \qquad \text{for some} \qquad m \in (0, r) \qquad \text{and all} \qquad t \in [a, a + \tau].$$

Then for all $t \in [a, a + \tau]$

$$\| x(t) - y(t)\| \leqslant [\| x(a) - y(a)\| + 2(\tau + 1) H_m(a)] e^{2L\tau}, \tag{5.1}$$

where, using Definition 2.18, $H_m(a) = \sup_{t \geqslant a} \int_t^{t+1} h_m(s)\, ds$.

Proof. Let $\lambda = \sup \{\| x(t) - y(t)\| : a \leqslant t \leqslant a + \tau\}$. Let $t \in [a, a + \tau]$. Then, using Definition 2.18,

$$\langle x'(t) - y'(t), x(t) - y(t)\rangle \leqslant L \| x(t) - y(t)\|^2 + \lambda h_m(t).$$

Integrating both sides, we have

$$\| x(t) - y(t)\|^2 \leqslant \| x(a) - y(a))\|^2 + 2\lambda(\tau + 1) H_m(a) + \int_a^t 2L \| x(s) - y(s)\|^2 \, ds.$$

By Lemma 2.1,

$$\| x(t) - y(t)\|^2 \leqslant [\| x(a) - y(a)\|^2 + 2\lambda(\tau + 1) H_m(a)] e^{2L\tau},$$

The proof can now be completed easily.

THEOREM 5.2. *For the class $\mathcal{F}_{\text{Inn}}$*

$$\mathcal{G}(\mathcal{F}_{\text{Inn}}) \supset \{g(t, x) : g \text{ is absolutely diminishing}\},$$

$$\mathcal{H}(\mathcal{F}_{\text{Inn}}) \supset \{h(t) : h \text{ is absolutely diminishing}\}.$$

Furthermore, $\mathcal{H}(\mathcal{F}_{\text{Inn}}) \subsetneqq \mathcal{H}(\mathcal{F}_{\text{Lip}})$ *and* $\mathcal{G}(\mathcal{F}_{\text{Inn}}) \subsetneqq \mathcal{G}(\mathcal{F}_{\text{Lip}})$.

Remark. Whether or not $\mathcal{H}(\mathcal{F}_{\text{Inn}})$ equals the class of absolutely diminishing functions is an open question.

Proof. By Lemma 2.11 and since

$$\langle x - y, f(t, x) - f(t, y)\rangle \leqslant \| x - y\| \| f(t, x) - f(t, y)\|,$$

we see that $\mathcal{G}(\mathcal{F}_{\text{Inn}}) \subset \mathcal{G}(\mathcal{F}_{\text{Lip}})$ and $\mathcal{H}(\mathcal{F}_{\text{Inn}}) \subset \mathcal{H}(\mathcal{F}_{\text{Lip}})$.

Let $f \in \mathcal{F}_{\text{Inn}}$, let 0 be EvUAS for (E), and let $g(t, x)$ be absolutely diminishing (with constant r). Shrink r if necessary so that f has an inner product constant L on $[0, \infty) \times S_r$. Let $\tau \geqslant 0$, $t_0 \geqslant \alpha(r)$, and $\| x_0 \| < \delta(r)$. Then $\| x(t; t_0, x_0)\| < r$ for all $t \geqslant t_0$ by uniform stability. Let $0 < m < r$. Then for any solution $y(\cdot)$ of (P) satisfying

$$m \leqslant | y(t; t_0, x_0)| \leqslant r \qquad \text{for} \qquad t_0 \leqslant t \leqslant t_0 + \tau_1,$$

where $\tau_1 \leqslant \tau$, we have by Lemma 5.1

$$\| x(t; t_0, x_0) - y(t; t_0, x_0)\| \leqslant 2(\tau + 1)\, e^{2L\tau} H_m(t_0).$$

Since m was arbitrary, the assumptions of Lemma 3.1 hold with $\lambda \equiv 0$ and

$$F_{\tau,m} = 2(\tau + 1)\, e^{2L\tau} H_m .$$

Hence 0 is EvUAS for (P). Now we need only show that $\mathcal{H}(\mathcal{F}_{\text{Inn}}) \neq \mathcal{H}(\mathcal{F}_{\text{Lip}})$

Example 5.3. Consider the linear system

$$x' = A(t)x, \qquad A(t) = \begin{pmatrix} -1 & e^t \\ -e^t & -1 \end{pmatrix}. \tag{5.2}$$

Then a fundamental matrix of (5.2) satisfies

$$X(t) = e^{-t} \begin{pmatrix} \sin e^t & -\cos e^t \\ \cos e^t & \sin e^t \end{pmatrix},$$

$$X^{-1}(t) = e^t \begin{pmatrix} \sin e^t & \cos e^t \\ -\cos e^t & \sin e^t \end{pmatrix}.$$

Then $| X(t)\, X^{-1}(s)| \leqslant Ke^{-(t-s)}$ for some constant $K > 0$ and all $t \geqslant s \geqslant 0$. Thus 0 is UAS for (5.2). Furthermore, $\langle x, A(t)x\rangle = -\| x \|^2$ for all $x \in R^2$, hence $A(t)x \in \mathcal{F}_{\text{Inn}}$. Let

$$h(t) = \begin{pmatrix} \sin e^t \\ \cos e^t \end{pmatrix}.$$

Then h is diminishing; in fact, for any $u \geqslant t \geqslant 0$,

$$\left| \int_t^u h(s)\, ds \right| \leqslant 6e^{-t}. \tag{5.3}$$

We also have

$$X(t) \int_{t_0}^t X^{-1}(s)\, h(s)\, ds = [1 - e^{-(t-t_0)}] \begin{pmatrix} \sin e^t \\ \cos e^t \end{pmatrix}.$$

Hence for any $t_0 \geqslant 0$, the solution of

$$y' = A(t)y + h(t) \tag{5.4}$$

through $(t_0, 0)$ satisfies

$$\| y(t; t_0, 0)\| = \| X(t) \int_{t_0}^{t} X^{-1}(s)\, h(s)\, ds \| \to 1 \qquad \text{as} \qquad t \to \infty.$$

Therefore $h \in \mathcal{H}(\mathcal{F}_{\mathrm{Lip}})$ but $h \notin \mathcal{H}(\mathcal{F}_{\mathrm{Inn}})$. This shows that $\mathcal{H}(\mathcal{F}_{\mathrm{Inn}}) \neq \mathcal{H}\mathcal{F}_{\mathrm{Lip}})$ and completes the proof of Theorem 5.2.

Remark. There exists a Liapunov function for (5.2), namely, $V(x_1, x_2) = x_1^2 + x_2^2$ for which $\dot{V}(x_1, x_2) = -2V(x_1, x_2)$. This shows that the existence of a very nice Liapunov function cannot *by itself* prove perturbation results in which $g(t, x)$ is diminishing.

Remark 5.4. Note that $f(t, x) = A(t)x$ in Example 5.3 is unbounded on $[0, \infty) \times S_1$. We think that this has to be the case, because we conjecture that if $\mathcal{F}_{\mathrm{BInn}}$ *is the class of bounded, inner product functions,* $\mathcal{G}(\mathcal{F}_{\mathrm{BInn}})$ *contains all diminishing functions.*

In § 4 we were able to show that the class $\mathcal{H}(\mathcal{F}_{\mathrm{Lip}})$ is a vector space over the field of real numbers. We did this by precisely identifying it. Although we have not been able to identify $\mathcal{H}(\mathcal{F}_{\mathrm{Inn}})$, and although we do not know whether it is a vector space, we can still prove

THEOREM 5.5. $\mathcal{H}(\mathcal{F}_{\mathrm{Inn}})$ *is closed under addition.*

Proof. $\mathcal{F}_{\mathrm{Inn}}$ is closed under addition. Thus, by Lemma 2.13, $\mathcal{G}(\mathcal{F}_{\mathrm{Inn}})$ is closed under the addition of two inner product functions. Since every member of $\mathcal{H}(\mathcal{F}_{\mathrm{Inn}})$ is an inner product function, it is closed under addition.

The following result will be useful later. Of course it is true also for the class $\mathcal{F}_{\mathrm{Lip}}$.

LEMMA 5.6. *Let* $f \in \mathcal{F}_{\mathrm{Inn}}$. *If* 0 *is* EvUA *for* (E), *then* 0 *is* EvUAS *for* (E).

Proof. Let 0 be EvUA for (E). Choose the corresponding $\delta_0 > 0$, $\alpha_0 \geqslant 0$, and $T(\cdot)$. Let $v(t)$ be a solution of (E), defined for $t \geqslant \alpha_0$, such that $\| v(t)\| \to 0$ as $t \to \infty$. (If we knew that $f(t, 0) \equiv 0$, we would choose $v(t) \equiv 0$.) Let $\epsilon > 0$. Shrink ϵ if necessary so that f has inner product constant L on $[0, \infty) \times S_\epsilon$. Choose $\delta = \delta(\epsilon) > 0$ so that

$$3\delta < \min\,(\epsilon e^{-2LT(\epsilon)}, \epsilon, 3\delta_0).$$

Choose $\alpha = \alpha(\epsilon) \geqslant \alpha_0$ so that $\| v(t)\| \leqslant \delta$ for all $t \geqslant \alpha$. Let $t_0 \geqslant \alpha$ and $\| x_0 \| < \delta$. Applying Lemma 5.1 with $g(t, x) \equiv 0$, we see that, if

$$\| x(t; t_0, x_0)\| < \epsilon \tag{5.5}$$

for $t_0 \leqslant t < t_1 \leqslant t_0 + T(\epsilon)$, then at $t = t_1$ we have

$$\begin{aligned} \| x(t_1 ; t_0 , x_0)\| &\leqslant \| x(t_1 ; t_0 , x_0) - v(t_1)\| + \| v(t_1)\| \\ &\leqslant \| x_0 - v(t_0)\| e^{2LT(\epsilon)} + \epsilon/3 \\ &\leqslant \| x_0 \| e^{2LT(\epsilon)} + \| v(t_0)\| e^{2LT(\epsilon)} + \epsilon/3 < \epsilon, \end{aligned}$$

using (5.1). But then $\| x(t; t_0 , x_0)\|$ cannot reach the value ϵ for $t \in [t_0 , t_0 + T(\epsilon)]$. Thus (5.5) holds for $t_0 \leqslant t \leqslant t_0 + T(\epsilon)$. For $t \geqslant t_0 + T(\epsilon)$, (5.5) holds since 0 is EvUA. Thus (5.5) holds for all $t \geqslant t_0 \geqslant \alpha(\epsilon)$ and $\| x_0 \| < \delta(\epsilon)$. Hence 0 is EvUS, and thus EvUAS, completing the proof.

We can now show that neither Corollary 4.5 nor Theorem 4.8 is true for inner product functions f.

Example 5.7. Consider the scalar equation

$$x' = \psi(t, x) = -2tx + 1, \tag{5.6}$$

so that $\psi \in \mathscr{F}_{\mathrm{Inn}}$. Then

$$x(t; t_0 , x_0) = e^{-t^2} e^{t_0^2} x_0 + e^{-t^2} \int_{t_0}^{t} e^{s^2} \, ds.$$

is the solution of (5.6) through (t_0 , x_0). It is not hard to show that 0 is EvUA for (5.6).
By Lemma 5.6, 0 is EvUAS for (5.6). Since 0 is clearly UAS for $x' = -2tx$, and since $\psi(t, 0) \equiv 1$ which is not diminishing, Lemma 4.1, Corollary 4.5, and Theorem 4.8 fail for inner product functions.

Since $\mathcal{G}(\mathscr{F}_{\mathrm{Inn}}) \subset \mathcal{G}(\mathscr{F}_{\mathrm{Lip}})$, it follows directly from Theorem 4.9 that

THEOREM 5.8. *If $d \geqslant 2$, then $\mathcal{G}(\mathscr{F}_{\mathrm{Inn}})$ contains no nontrivial, Lipschitz function which is independent of t.*

6. LINEAR FUNCTIONS

Define $\mathscr{F}_{\mathrm{Lin}} = \{f \in \mathscr{F}_C : f(t, x) = A(t)x\}$. For each function $f \in \mathscr{F}_{\mathrm{Lin}}$, 0 is UAS for (E) if and only if 0 is EvUAS for (E). We continue to use EvUAS for easy comparison with previous results. If $k \in \mathscr{F}_C$ then we shall write $k = o(| x |)$ if

$$\lim_{\substack{|x| \to 0 \\ t \to \infty}} | x |^{-1} | k(t, x)| = 0.$$

THEOREM 6.1. *For the class* $\mathcal{F}_{\text{Lin}}$

$$\mathcal{G}(\mathcal{F}_{\text{Lin}}) \supset \{g(t, x) + k(t, x) : g \text{ is absolutely diminishing and } k = o(|x|)\},$$
$$\mathcal{H}(\mathcal{F}_{\text{Lin}}) \supset \{h(t) : h \text{ is absolutely diminishing}\}.$$

Furthermore, $\mathcal{H}(\mathcal{F}_{\text{Lin}}) \subsetneqq \mathcal{H}(\mathcal{F}_{\text{Lip}})$ *but* $\mathcal{G}(\mathcal{F}_{\text{Lin}})$ *and* $\mathcal{G}(\mathcal{F}_{\text{Lip}})$ *are not related by inclusion.*

As in § 5, whether or not $\mathcal{H}(\mathcal{F}_{\text{Lin}})$ equals the class of absolutely diminishing functions is an open question.

Proof. Fix $A(t)x$ in $\mathcal{F}_{\text{Lin}}$ and consider

$$x' = A(t)x. \tag{6.1}$$

Let (6.1) have fundamental matrix $X(t)$ and let 0 be EvUAS for (6.1). Then we are given $\alpha(\cdot)$ and $\delta(\cdot)$ and by ([3], p. 54) there exists $K \geqslant 1$ and $\sigma > 0$ such that

$$|X(t)X^{-1}(s)| \leqslant Ke^{-\sigma(t-s)} \qquad \text{for all} \qquad t \geqslant s \geqslant 0.$$

Let $k = o(|x|)$. Let $0 < \eta < \sigma K^{-1}$. Let g be absolutely diminishing for some $r > 0$. If necessary. shrink r and increase $\alpha(r)$ so that $|k(t, x)| \leqslant \eta |x|$ for all $t \geqslant \alpha(r)$ and $|x| \leqslant r$. Let $0 < m < r$. Let $t_0 \geqslant \alpha(r)$, $|x_0| < \delta(r)$, and $\tau > 0$. Let $y(t) = y(t; t_0, x_0)$ be any solution of

$$y' = A(t)y + k(t, y) + g(t, y). \tag{6.2}$$

Let $\tau_1 \leqslant \tau$ and suppose $m \leqslant |y(t; t_0, x_0)| \leqslant r$ for $t_0 \leqslant t \leqslant t_0 + \tau_1$. Let $\beta = \sigma - K\eta$. Then

$$|y(t)| = |X(t)X^{-1}(t_0)x_0 + \int_{t_0}^{t} X(t)X^{-1}(s)[k(s, y(s)) + g(s, y(s))]\,ds|$$
$$\leqslant Ke^{-\sigma t}e^{\sigma t_0}|x_0| + Ke^{-\sigma t}\int_{t_0}^{t} e^{\sigma s}[n|y(s)| + h_m(s)]\,ds.$$

Multiplying both sides by $e^{\sigma t}$, applying Lemma 2.1 to the function $r(t) = e^{\sigma t}|y(t)|$, then multiplying by $e^{-\sigma t}$, we obtain

$$|y(t)| \leqslant Ke^{-\beta(t-t_0)}|x_0| + Ke^{-\beta t}\int_{t_0}^{t} e^{\beta s}h_m(s)\,ds$$
$$\leqslant Ke^{-\beta(t-t_0)}|x_0| + K(\tau + 1)H_m(t_0),$$

where

$$H_m(t) = \sup_{T \geqslant t}\int_{T}^{T+1} h_m(s)\,ds$$

Hence the assumptions of Lemma 3.1 hold with $x(t) \equiv 0$, $\lambda(\rho) = K\rho$, $\nu(s) = e^{-\beta s}$, and $F_{\tau,m} = K(\tau + 1)H_m$. Thus 0 is EvUAS for (6.2) by Lemma 3.1. Hence $g + k$ belongs to $\mathcal{G}(\mathcal{F}_{\text{Lin}})$.

The function $h(t)$ in Example 5.3 belongs to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$ and not to $\mathcal{G}(\mathcal{F}_{\mathrm{Lin}})$ because equation (5.2) is linear. The function

$$g(t, x) = (x^2, 0,..., 0) = o(|x|)$$

belongs to $\mathcal{G}(\mathcal{F}_{\mathrm{Lin}})$ but not to $\mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$ by Theorem 4.9. Thus

$$\mathcal{G}(\mathcal{F}_{\mathrm{Lin}}) \neq \mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$$

and neither class contains the other.

Finally, $\mathcal{H}(\mathcal{F}_{\mathrm{Lin}}) \subset \mathcal{H}(\mathcal{F}_{\mathrm{Lip}})$ using Corollary 4.5 and the linear, Lipschitz function $f(t, x) = -x$, while $\mathcal{H}(\mathcal{F}_{\mathrm{Lin}}) \neq \mathcal{H}(\mathcal{F}_{\mathrm{Lip}})$ by Example 5.3. This completes the proof.

For the case $A(t) \equiv A$, a constant matrix, 0 is EvUAS for

$$x' = Ax \tag{6.3}$$

if and only if every eigenvalue of A has negative real part. That (6.3) may be perturbed by a function $k = o(|x|)$ has been known for a long time (for example, see [*2*], p. 314). Coddington and Levinson ([*2*], p. 327) showed that if $k = o(|x|)$ and, for some $r > 0$,

$$\sup_{|x| \leqslant r} |g(t, x)| \to 0 \qquad \text{as} \qquad t \to \infty, \tag{6.4}$$

then all solutions of

$$x' = Ax + k(t, x) + g(t, x) \tag{6.5}$$

which have sufficiently small $|x_0|$ and sufficiently large t_0 approach zero as $t \to \infty$. Brauer [*1*] improved their result to the case where g satisfied either (6.4) or $|g(t, x)| \leqslant \lambda(t)|x|$ for $t \geqslant 0$ and $|x| \leqslant r$ and $\int_0^\infty \lambda(t)\, dt < \infty$. The authors [*11*] extended Brauer's result to the case where g is absolutely diminishing for $m = 0$. Although the proofs of these results appear to work also for the case of variable $A(t)$, none of the proofs establish the uniformity of the asymptotic stability of 0 for (6.5). While it is true that the proof of Theorem 3.2 in [*11*] can be extended to give the EvUAS of 0 for (6.5), and while this extension is fairly straightforward when g is absolutely diminishing for $m = 0$, this extension is very long and involved when g is merely absolutely diminishing. In fact this extension turns out to essentially imitate the proof of Lemma 3.1. Therefore, since Lemma 3.1 was available, we were able to give a reasonably efficient proof, however unnatural it may have seemed, of Theorem 6.1. Other results on perturbed linear systems usually are proved for bounded matrices $A(t)$ (for example, see ([*15*], p. 121). If $A(t)$ were bounded, however, then $A(t)x \in \mathcal{F}_{\mathrm{Lip}}$ and Theorem 4.4 shows that we could perturb (6.1) by any diminishing function. In the general situation, of course, we cannot, as Example 5.3 shows.

As was the case in §5, Corollary 4.5 breaks down for general linear systems because Example 5.7 is linear. Also, as was the case in §5, we cannot identify $\mathcal{H}(\mathcal{F}_{\mathrm{Lin}})$. We can, however, show a little more than we could in Theorem 5.5.

THEOREM 6.2. *$\mathcal{H}(\mathcal{F}_{\mathrm{Lin}})$ is a vector space over the real numbers.*

Proof. Fix $A(t)x \in \mathcal{F}_{\mathrm{Lin}}$. Let 0 be EvUAS for (E). From the variation of constants formula applied to $x' = A(t)x + h(t)$ and from Lemma 2.9, we see that for this *fixed* function $A(t)x$, $\mathcal{H}(\{A(t)x\})$ is a vector space over the reals. By Lemma 2.12, $\mathcal{H}(\mathcal{F}_{\mathrm{Lin}})$ is the intersection of vector spaces, and hence it is a vector space.

In the linear case, however, Lemma 5.6 is false, as can be seen from an example due to Massera [*8*]; see also ([*12*], Example 4.4).

7. PERIODIC FUNCTIONS

We say that f is a *periodic* function if there exists $\omega > 0$ such that $f(t + \omega, x) = f(t, x)$ for all $t \geqslant 0$ and $x \in R^d$. Define

$$\mathcal{F}_{\mathrm{Per}} = \{f \in \mathcal{F}_c : f \text{ is periodic}\}.$$

Note that $\mathcal{F}_{\mathrm{Per}}$ contains the continuous functions which are independent of t, that we do not assume f is locally Lipschitz, and that we do not assume that (E) has uniqueness to the right for $f \in \mathcal{F}_{\mathrm{Per}}$.

THEOREM 7.1. *For the class $\mathcal{F}_{\mathrm{Per}}$*

$$\mathcal{G}(\mathcal{F}_{\mathrm{Per}}) \supset \{g(t, x) : g \textit{ is diminishing}\}$$

$$\mathcal{H}(\mathcal{F}_{\mathrm{Per}}) = \mathcal{H}(\mathcal{F}_{\mathrm{Lip}}).$$

Proof. Let f be continuous and periodic with least period $\omega > 0$. If f is independent of t, we choose $\omega = 1$. Let g be diminishing with constant $r > 0$. Define

$$E_m{}^*(t) = \sup \left| \int_t^{t+u} g(s, y(s))\, ds \right|,$$

where the supremum is evaluated with respect to all solutions $y(\cdot)$ of (P) satisfying

$$m \leqslant |y(s)| \leqslant r \qquad \text{for} \qquad t \leqslant s \leqslant t + u$$

and then with respect to all $0 \leqslant u \leqslant 1$. Then $E_m{}^*(t) \to 0$ as $t \to \infty$ just as in the proof of Theorem 4.4, except that now one uses the fact that f

is bounded on $[0, \infty) \times S_r$ (instead of the fact that f is Lipschitz there and $f(t, 0)$ is diminishing).

Define

$$E_m(t) = \sup_{T \geqslant t} E_m{}^*(T).$$

Then $E_m(t) \downarrow 0$ as $t \to \infty$ for every $m \in (0, r)$. Let 0 be EvUAS for (E). We shall apply Lemma 3.2 with the constant $r/2$. Suppose the hypotheses of Lemma 3.2 do not hold for $r/2$. Then there exist numbers $m \in (0, r/2)$, $\tau > 0$, and $\epsilon > 0$; sequences of numbers $t_n \to \infty$ and $\sigma_n \in (0, \tau]$; and a sequence of solutions $y_n(\cdot)$ of (P) satisfying $| y_n(t_n)| < \delta(r/2)$ and

$$m \leqslant | y_n(t)| \leqslant r/2 \qquad \text{for} \qquad t_n \leqslant t \leqslant t_n + \sigma_n$$

such that for every solution $x(\cdot)$ of (E),

$$| x(u) - y_n(u)| \geqslant \epsilon \tag{7.1}$$

for at least one u in $[t_n, t_n + \sigma_n]$. Our goal now is to arrive at a contradiction.

Let $\omega_n \geqslant 0$ be that unique integral multiple of ω such that the number $\hat{t}_n = t_n - \omega_n$ belongs to $[0, \omega)$. By choosing a subsequence if necessary, we may assume that $\hat{t}_n \to t \in [0, \omega]$ and $\sigma_n \to \sigma \in [0, \tau]$. Define

$$\phi_n(t) = y_n(t + \omega_n) \qquad \text{for all} \qquad \hat{t}_n \leqslant t \leqslant \hat{t}_n + \sigma_n .$$

Then for such t, $m \leqslant | \phi_n(t)| \leqslant r/2$ and

$$\phi_n(t) = \phi_n(\hat{t}_n) + \int_{\hat{t}_n}^{t} f(s, \phi_n(s))\, ds + \int_{\hat{t}_n+\omega_n}^{t+\omega_n} g(s, \phi_n(s - \omega_n))\, ds, \tag{7.2}$$

since f has period ω.

We first show that for some $\delta > 0$ and for all sufficiently large n, we have

$$m/2 \leqslant | \phi_n(t)| \leqslant r \qquad \text{for all} \qquad \hat{t} - \delta \leqslant t \leqslant \hat{t} + \sigma + \delta. \tag{7.3}$$

Choose ζ so that $0 < \zeta < m/2 < r/2$. Then $|f(t, x)| \leqslant M$ for some M and for all $t \geqslant 0$ and $x \in S_r$. Choose N so large that

$$E_{m/2}(t_N - 1) < \zeta/2$$

and choose δ so that $0 < \delta < 1$ and $4M\delta < \zeta$. Let $n > N$ and $| t - \hat{t} | < \delta$. Then for as long as $m/2 \leqslant | \phi_n(t)| \leqslant r$, we have from (7.2)

$$| \phi_n(t) - \phi_n(\hat{t}_n)| \leqslant M \mid t - \hat{t}_n \mid + E_{m/2}(t_n) < \zeta.$$

Thus $\phi_n(\cdot)$ exists and is bounded between $m/2$ and r on $[\hat{t} - \delta, \hat{t} + \delta]$ .A similar argument works for the interval $[\hat{t} + \sigma - \delta, \hat{t} + \sigma + \delta]$. Thus $\phi_n(\cdot)$ satisfies (7.3). Let

$$G_n(t) = \int_{\hat{t}_n+\omega_n}^{t+\omega_n} g(s, \phi_n(s - \omega_n))\, ds.$$

Then if t' and t'' belong to $[\hat{t} - \delta, \hat{t} + \sigma + \delta]$,

$$|[\phi_n(t') - G_n(t')] - [\phi_n(t'') - G_n(t'')]| \leqslant M \mid t' - t'' \mid.$$

Therefore the sequence $\{\phi_n - G_n\}$ is uniformly bounded and equicontinuous on $[\hat{t} - \delta, \hat{t} + \sigma + \delta]$. Thus this sequence has a subsequence $\{\phi_{n_j} - G_{n_j}\}$ which converges uniformly on this interval to a continuous function $\phi(\cdot)$. Relabel so that this subsequence is written $\{\phi_n - G_n\}$. Since

$$| G_n(t)| \leqslant (\sigma + 2)\, E_{m/2}(t_n - 1),$$

$\phi_n \to \phi$ uniformly on $[\hat{t} - \delta, \hat{t} + \sigma + \delta]$. By (7.2),

$$\phi(t) = \sigma(\hat{t}) + \int_{\hat{t}}^{t} f(s, \phi(s))\, ds.$$

Thus $\phi(\cdot)$ is a solution of (E) on $[\hat{t} - \delta, \hat{t} + \sigma + \delta]$. Define

$$\psi_n(t) = \phi(t - \omega_n) \qquad \text{for} \qquad t \in I_n = [\hat{t} + \omega_n - \delta, \hat{t} + \sigma + \omega_n + \delta].$$

Then $\psi_n(\cdot)$ is a solution of (E) for each n and $| \psi_n(t) - y_n(t)| \to 0$ as $n \to \infty$ uniformly for $t \in I_n$. Since $[t_n, t_n + \sigma_n] \subset I_n$ for sufficiently large n, we have contradicted (7.1). Thus the hypotheses of Lemma 3.2 hold. By Lemma 3.2, 0 is EvUAS for (P). Thus $\mathcal{G}(\mathcal{F}_{\mathrm{Per}})$ and $\mathcal{H}(\mathcal{F}_{\mathrm{Per}})$ contain all diminishing functions.

The function $f(t, x) = -x$ belongs to $\mathcal{F}_{\mathrm{Per}}$ and to $\mathcal{F}_{\mathrm{Lip}}$. By Corollary 4.5 and Lemma 2.12, $\mathcal{H}(\mathcal{F}_{\mathrm{Per}}) \subset \mathcal{H}(\mathcal{F}_{\mathrm{Lip}})$. Thus $\mathcal{H}(\mathcal{F}_{\mathrm{Per}}) = \mathcal{H}(\mathcal{F}_{\mathrm{Lip}})$ and the proof of Theorem 7.1 is complete.

In Theorem 4.9 we showed that if g is any nontrivial Lipschitz function which is independent of t, then $g \notin \mathcal{G}(\mathcal{F}_{\mathrm{Lip}})$. We did this by constructing f such that $f \in \mathcal{F}_{\mathrm{Lip}}$, 0 is EvUAS for (E) and 0 is not EvUAS for (P). In that construction f was chosen independent of t, so that $f \in \mathcal{F}_{\mathrm{Per}}$. Since, in this section, we do not need f to be Lipschitz, the proof of Theorem 4.9 for not-necessarily-Lipschitz g proves

THEOREM 7.2. *If $d \geqslant 2$, then $\mathcal{G}(\mathcal{F}_{\mathrm{Per}})$ contains no nontrivial continuous function which is independent of t.*

In the following analog of Lemma 5.6, we do not assume *a priori* that 0 is a solution, though as the proof shows, 0 is a unique-to-the-right solution.

LEMMA 7.3. *Let $f \in \mathcal{F}_{\mathrm{Per}}$. If* 0 *is* EvUA *for* (E), *then* 0 *is* UAS *for* (E).

Proof. Let $v(t)$ be a solution of (E) which tends to zero as $t \to \infty$. Let $\omega > 0$ be a period of f. Then for each $t \geqslant 0$,

$$v(t + n\omega) \to 0 \qquad \text{as} \qquad n \to \infty. \tag{7.4}$$

Now $v(t + n\omega) = v(t + n\omega; n\omega, v(n\omega))$ is some solution of (E) through $(0, v(n\omega))$, call it $x(t; 0, v(n\omega))$. For each fixed $\tau \geqslant 0$, there exists a sequence n_k of integers such that

$$x(t; 0, v(n_k\omega)) \to x(t; 0, 0) \qquad \text{as} \qquad n_k \to \infty$$

uniformly for $t \in [0, \tau]$, where $x(t, 0, 0)$ denotes some solution through (0,0). But this means

$$v(t + n_k\omega) \to x(t; 0, 0) \qquad \text{as} \qquad n_k \to \infty$$

for $t \in [0, \tau]$. By (7.4), $x(t; 0, 0) \equiv 0$ for $0 \leqslant t \leqslant \tau$. Since τ is arbitrary, the zero function is a solution of (E) on $[0, \infty)$. We now show it is unique to the right.

Let $y(t; 0, 0)$ denote any solution of (E) through $(0, 0)$. Define for $n = 1, 2, \ldots$,

$$y_n(t; 0, 0) = \begin{cases} 0 & \text{if } \; 0 \leqslant t \leqslant n\omega \\ y(t - n\omega; 0, 0) & \text{if } \; n\omega \leqslant t. \end{cases}$$

Since the zero function is a solution of (E), so is $y_n(t; 0, 0)$ for each n. Let $\epsilon > 0$. Choose $T = T(\epsilon) \geqslant 0$ by Definition 2.3. Choose $N = N(\epsilon)$ so large that $N\omega \geqslant \alpha_0 + T$. Let $t \geqslant N\omega$. Then

$$| y_N(t; 0, 0)| = | y(t - N\omega; 0, 0)| < \epsilon$$

for all $t \geqslant N\omega$. This means $| y(t; 0, 0)| < \epsilon$ for all $t \geqslant 0$. Since $\epsilon > 0$ was arbitrary, $y(t; 0, 0) = 0$ for all $t \geqslant 0$. Thus the zero function is a unique-to-the-right solution of (E). By Lemma 2.6, 0 is UA for (E). By continuous dependence arguments, 0 is also US for (E), completing the proof.

Corollary 4.5 holds for periodic functions as we now show.

Corollary 7.4. *If $f \in \mathcal{F}_{\text{Per}}$ and* 0 *is* EvUAS *for* (E), *then* 0 *is* EvUAS *for*

$$y' = f(t, y) + h(t) \tag{7.5}$$

if and only if h is diminishing.

Proof. The "if" part follows from Theorem 7.1. Suppose 0 is EvUAS for (E) and (7.5). By Lemma 7.3, $f(t, 0) = 0$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $\delta < \epsilon$ and $| f(t, y)| < \epsilon$ for all $t \geqslant 0$ and $| y | < \delta$. Let $y(\cdot)$ be a solution of (7.5) which tends to zero as $t \to \infty$. Choose $\tau = \tau(\epsilon)$ so large that $| y(t)| < \delta$ for all $t \geqslant \tau$. Then if $\tau \leqslant t$ and $0 \leqslant u \leqslant 1$, we have

$$\left| \int_t^{t+u} h(s)\, ds \right| = | y(t + u) - y(t) - \int_t^{t+u} f(s, y(s))\, ds | < 3\epsilon.$$

Thus h is diminishing, completing the proof.

8. Continuous Functions: Some Examples

The main purpose of this section is to show that $\mathcal{G}(\mathcal{F}_c)$ and $\mathcal{K}(\mathcal{F}_c)$ do not contain all the absolutely diminishing functions.

Example 8.1. We first show that if, in the case of non-unique solutions, we were to say that 0 is EvUAS when *at least one solution* through each point satisfied Definition 2.4, then we would not be able to perturb certain autonomous systems at all, in contrast to the results in §7. Define $\phi : [-1, 1] \to R^1$ by

$$\phi(z) = \begin{cases} -2\pi(2^{-n} - z)^{1/2} (z - 2^{-n-1})^{1/2} & \text{if } 2^{-n-1} \leqslant z \leqslant 2^{-n}, \quad n = 0, 1,..., \\ -z & \text{if } -1 \leqslant z \leqslant 0. \end{cases}$$

Consider the scalar equation

$$z' = \phi(z). \tag{8.1}$$

For each $n = 0, 1,...,$ and each $t_0 \geqslant 0$, there are many solutions through $(t_0, 2^{-n})$, one of which is $z(t) \equiv 2^{-n}$ and another of which is

$$z(t) = 2^{-n} - 2^{-n-1} \sin^2 \pi(t - t_0)$$

for $t_0 \leqslant t \leqslant t_0 + 1/2$. For $x_0 > 0$ let $z^*(t; t_0; x_0)$ denote that solution of (8.1) through (t_0, x_0) which is strictly decreasing for all $t \geqslant t_0$. Then for example,

$$z^*(t; t_0, 1) = 2^{-n} - 2^{-n-1} \sin^2 \pi(t - t_0 - n/2)$$

for $t_0 + n/2 \leqslant t \leqslant t_0 + (n + 1)/2$. For $0 < \xi \leqslant 1$, let $n(\xi)$ be the smallest nonnegative integer such that $\xi \leqslant 2^{-n(\xi)}$. Then a straight forward computation shows that, for $0 < x_0 \leqslant 1$,

$$z^*(t; t_0, x_0) \leqslant z^*(t; t_0, 2^{-n(x_0)}) \leqslant 2^{-n(x_0)+1}2^{-(t-t_0)} \leqslant 4x_0 2^{-(t-t_0)}. \tag{8.2}$$

Thus 0 is *weakly exponentially stable* for (8.1), i.e., there exist $k \geqslant 1$ and $\sigma > 0$ such that for all $t_0 \geqslant 0$ and $|x_0| \leqslant 1$,

$$|z(t; t_0, x_0)| \leqslant k |x_0| e^{-\sigma(t-t_0)}$$

for at least one solution through (t_0, x_0). Furthermore, (8.1) is autonomous, ϕ is continuous, and $|\phi(z)| \leqslant 2\pi |z|$ for all $|z| \leqslant 1$. Nevertheless, 0 does not have any kind of (weak) attraction property for

$$y' = \phi(y) + g(t, y)$$

if $g(t, y) > 0$ when $y > 0$. This example shows that in Theorem 7.1 it was important that *all* solutions of (E) behaved properly.

Example 8.2. We now show that, even for exponential stability, there exists a locally Lipschitz, uniformly continuous function which cannot be perturbed by such nice functions as $x^n e^{-nt}$, $n = 0, 1, 2, \ldots$

Let ϕ be as in Example 8.1 and define

$$f(t, x) = \begin{cases} \min\{\phi(x), -xe^{-t^2}\} & \text{if} \quad 0 < x < 3/4 \\ -2\pi x/3 & \text{if} \quad x \leqslant 0 \text{ or if } x \geqslant 3/4. \end{cases}$$

Then f is uniformly continuous on $[0, \infty) \times R^1$, $|f(t, x)| \leqslant 2\pi |x|$ for all real x, and a long calculation establishes that f is locally Lipschitz; in fact, for some $c > 0$,

$$|f(t, x) - f(t, y)| \leqslant ce^t |x - y|$$

for all $t \geqslant 0$, x real, and y real. It follows from standard comparison theorems and from (8.2) that the solutions of

$$x' = f(t, x) \tag{8.3}$$

satisfy, for $0 < x_0 < 3/4$,

$$x(t; t_0, x_0) \leqslant z^*(t; t_0, x_0) \leqslant 4x_0 2^{-(t-t_0)}.$$

Therefore 0 is *globally exponentially stable*, in fact,

$$|x(t; t_0, x_0)| \leqslant 4 |x_0| 2^{-(t-t_0)} \qquad \text{for} \qquad t \geqslant t_0 \geqslant 0 \qquad \text{and all real} \qquad x_0,$$

which implies that 0 is UAS. Nevertheless, for every $\beta \geqslant 0$ and every continuous γ satisfying $\gamma(y) > 0$ for $y > 0$, 0 is not attracting for $y' = f(t, y) + \gamma(y) e^{-\beta t}$, because its right-hand side is positive for large t and $y = 2^{-n}$. This example shows that the various conditions on f in §4–7 cannot be weakened very much. This example also establishes Theorem C of §1.

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